The thermal boundary layer on a non-isothermal surface with non-uniform free stream velocity

By E. M. SPARROW
N.A.C.A. Lewis Flight Propulsion Laboratory, Cleveland, Ohio

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SUMMARY

A formally exact solution for the thermal boundary layer on a non-isothermal surface subjected to non-uniform free stream velocity is presented in the form of a series. It is demonstrated that the solution can be recast in terms of universal functions, which are independent of the wall temperature data of particular problems, and which depend only on a single parameter characterizing the variation of the free stream velocity.

INTRODUCTION

A new method for computing steady laminar incompressible boundary layer flows under conditions of non-uniform free stream velocity has recently been formulated by Görtler (1957). Provided that the free stream velocity can be written in terms of rather general series (or polynomials), he is able to give a formally exact solution for the boundary layer velocities. Further, he shows that the solution can be written in terms of universal functions, which depend only on a single parameter of the free stream flow.

Our interest here is in the thermal boundary layer, and we consider the general situation of a non-isothermal surface with non-uniform free stream velocity. Görtler's work will constitute a point of departure for the present study, and we will be able to make direct use of his results.

For situations where variations in both the surface temperature and free stream velocity are expressed as certain rather general series (with arbitrary coefficients), we are able to give a formally exact solution for the boundary layer temperature distribution. Our final results will involve a new set of universal functions. When these have been determined, the computation of the heat transfer and of other thermal quantities is a simple algebraic process.

Exact solutions of the boundary layer energy equation for non-isothermal surfaces have previously been found only for free stream velocity variations in which $U \propto x^p$. These are the so-called similar solutions. The present formulation permits the consideration of much more general variations of free stream velocity.

The development is carried out for steady laminar flow with constant fluid properties. Later, modifications for variable fluid properties will be noted.

The energy equation and its transformation

The differential equation expressing conservation of energy for steady laminar non-dissipative flow in a boundary layer is, in the usual notation (α being the thermal diffusivity),

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}. \tag{1}$$

The boundary conditions for the temperature are assigned as

$$T(x,0)=T_w(x), \tag{2a}$$

$$T(x,y)\to T_\infty \quad \text{as } y\sqrt{(Re)}\to\infty, \tag{2b}$$

where T_∞ is a constant. The terms describing the frictional heating and compression work have been omitted from the energy equation, thus restricting the problem to free stream velocities for which the adiabatic temperature rise is much smaller than the temperature difference between surface and stream.

An approach to the solution of equation (1) cannot be made without a knowledge of the velocity components u and v. A consequence of the constancy of the fluid properties is that a complete solution for the velocity may be made without recourse to the temperature. This same desirable independence of the velocity is also found for a special group of variations in fluid properties to be noted later. So, in relation to equation (1), the velocity components u and v can be regarded as known *a priori*. Since our goal is to attack as broad a problem as possible, we will select the most general velocity solutions presently available: namely, those of Görtler (1957).

Following Görtler, we introduce new independent variables ξ and η and a new dependent variable F by the relations

$$\xi=\frac{1}{\nu}\int_0^x U\,dx, \qquad \eta=yU\left\{2\nu\int_0^x U\,dx\right\}^{-1/2}, \tag{3}$$

$$F=\psi/\nu\sqrt{(2\xi)}, \tag{4}$$

where ψ is the stream function. In terms of these variables, the velocities u and v are

$$u=UF_\eta, \qquad v=-U(2\xi)^{-1/2}[F+2\xi F_\xi+(\beta-1)\eta F_\eta], \tag{5}$$

where the subscripts denote differentiation and β, termed the principal function by Görtler, is given by

$$\beta=\frac{2U'}{U^2}\int_0^x U\,dx. \tag{6}$$

For the temperature problem, we introduce a dimensionless variable Θ by the definition

$$\Theta\equiv\frac{T-T_\infty}{T_w-T_\infty}\equiv\frac{T-T_\infty}{\Delta T}. \tag{7}$$

Using the new independent and dependent variables, the energy equation (1) and the boundary conditions (2) may be written as

$$(Pr)^{-1}\Theta_{\eta\eta}+(F+2\xi F_{\xi})\Theta_{\eta} = 2\xi F_{\eta}\left[\Theta\left(\frac{\nu T'_w}{U\Delta T}\right)+\Theta_{\xi}\right], \tag{8}$$

$$\Theta(\xi,0) = 1, \qquad \Theta(\xi,\eta)\to 0 \quad \text{as } \eta\to\infty, \tag{9}$$

where T'_w is an abbreviation for dT_w/dx and Pr is the Prandtl number. All the given data concerning wall temperature and free stream velocity in any particular application appear explicitly only as the single coefficient $\nu T'_w/U\Delta T$, which we will call the principal thermal function and denote by λ. It is interesting to note that λ arises in the temperature problem in a manner similar to that in which β arises in the velocity problem. Both λ and β will play important roles in the solution of the temperature problem.

Görtler's solution for the velocity

As a final prelude to the solution of equation (8), the required information for the velocity variable F must be given. We choose to study situations where the free stream velocity may be represented by the convergent series

$$U(x) = x^m\{s_0+s_1x^{m+1}+s_2x^{2(m+1)}+\dots\}, \quad -1 < m < \infty, \quad s_0 \neq 0. \tag{10}$$

Included here are two very interesting physical situations. First, the case $m = 0$ represents flows with a forward cuspidal point at $x = 0$; and in particular, the flow over a flat plate is given by $s_1 = s_2 = \dots = 0$. Secondly, the case $m = 1$ represents flows about a symmetrical body* with a forward stagnation point.

The form of the principal function β, which corresponds to the free stream in equation (10) is

$$\beta = \beta_0+\beta_1\xi+\beta_2\xi^2+\dots, \tag{11}$$

where $\beta_0 = 2m/(m+1)$ and the other β_n depend on the coefficients s_n of equation (10). Further, the relationship connecting the coordinate x with ξ is found from equations (3) and (10) to be

$$\xi = \frac{x^{m+1}}{\nu(m+1)}\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n(m+1)}. \tag{12}$$

Finally, the solution for F is given by Görtler as

$$F(\xi,\eta) = \sum_{n=0}^{\infty}F_n(\eta)\xi^n, \tag{13}$$

where in turn the F_n are written as the following linear combinations of universal functions

$$\left.\begin{aligned} F_1 &= \beta_1 f_1,\\ F_2 &= \beta_1^2 f_{11}+\beta_2 f_2,\\ F_3 &= \beta_1^3 f_{111}+\beta_1\beta_2 f_{12}+\beta_3 f_3,\\ F_4 &= \beta_1^4 f_{1111}+\beta_1^2\beta_2 f_{112}+\beta_1\beta_3 f_{13}+\beta_2^2 f_{22}+\beta_4 f_4,\\ &\dots\dots\dots\dots \end{aligned}\right\} \tag{14}$$

* As pointed out by Görtler, the restriction of symmetry can be removed by consideration of an alternate series for the free stream velocity.

The function F_0 is determined from the Falkner–Skan differential equation containing β_0 as a parameter, while f_1, f_{11}, f_2, etc., are found from the solution of linear inhomogeneous ordinary differential equations with homogeneous boundary conditions. The important fact is that the equations for the functions f do not contain $\beta_1, \beta_2, \ldots$. Only β_0 enters indirectly through the appearance of F_0 and its derivatives. So, for a selected β_0, all the functions f as well as F_0 can be computed once and for all, without further reference to the additional special data ($\beta_1, \beta_2, \ldots$) of the particular problem. It is in this sense that the functions f are universal.

Series solution of the energy equation

We seek a solution of the energy equation (8) in the form of a series as follows:

$$\Theta = \sum_{n=0}^{\infty} \theta_n(\eta)\xi^n. \tag{15}$$

We will assume that the principal thermal function λ may also be expressed as a series:

$$\lambda = \sum_{n=0}^{\infty} \lambda_n \xi^n. \tag{16}$$

It will be shown in a later section that the dependence of the wall temperature on x which corresponds to this form for λ is a rather general series expansion. Further, we will show how to determine the coefficients λ_n for prescribed variations of wall temperature and of free stream velocity.

Introducing the series (13), (15) and (16) into the energy equation and grouping terms having common powers of ξ, we find a system of ordinary differential equations for the θ_n:

$$(Pr)^{-1}\theta_0'' + F_0\theta_0' = 0, \quad \theta_0(0) = 1, \quad \theta_0(\infty) = 0, \tag{17}$$

$$(Pr)^{-1}\theta_n'' + F_0\theta_n' - 2nF_0'\theta_n = \phi_n, \quad \theta_n(0) = \theta_n(\infty) = 0, \quad (n = 1, 2, \ldots), \tag{18}$$

where

$$\phi_n = \sum_{p=1}^{n} [2(n-p)F_p'\theta_{n-p} - (2p+1)F_p\theta_{n-p}'] + 2\sum_{p=0}^{n-1}\lambda_{n-p-1}\left(\sum_{j=0}^{p} F_j'\theta_{p-j}\right).$$

If we introduce the abbreviation

$$M_n(y) = (Pr)^{-1}y'' + F_0 y' - 2nF_0'y \tag{19}$$

then the first few of equations (18) are

$$M_1(\theta_1) = -3F_1\theta_0' + 2\lambda_0 F_0'\theta_0, \tag{20 a}$$

$$M_2(\theta_2) = 2F_1'\theta_1 - 3F_1\theta_1' - 5F_2\theta_0' + 2\lambda_0(F_1'\theta_0 + F_0'\theta_1) + 2\lambda_1 F_0'\theta_0, \tag{20 b}$$

$$M_3(\theta_3) = 4F_1'\theta_2 - 3F_1\theta_2' + 2F_2'\theta_1 - 5F_2\theta_1' - 7F_3\theta_0' + \\ + 2\lambda_0(F_2'\theta_0 + F_1'\theta_1 + F_0'\theta_2) + 2\lambda_1(F_1'\theta_0 + F_0'\theta_1) + 2\lambda_2 F_0'\theta_0. \tag{20 c}$$

Equations (17) and (18) constitute a recursive system for computation of the θ_n. All of equations (18) are linear. The boundary condition at the wall ($\eta = 0$) is satisfied by θ_0 alone, It is interesting to inspect the function ϕ_n, which contains the input data for the computation of θ_n. It is seen that the particular data embodied in $\beta_0, \ldots, \beta_n, \lambda_0, \ldots, \lambda_{n-1}$, and the Prandtl number, must be specified before equation (18) can be solved for θ_n. It is desirable to rephrase our problem in terms of functions which are independent of the explicit data of a particular problem. Such a course is followed below, where we will find universal functions which are completely independent of particular wall temperature conditions and which depend on the free stream velocity data only inasmuch as they depend on β_0.

The universal functions

Our aim is to reduce the functions θ_n to a linear combination of universal functions. First, we consider θ_1 and its differential equation (20 a). The velocity functions appearing on the right-hand side may be replaced by universal velocity functions (14) to give

$$M_1(\theta_1) = -\beta_1(3f_1\theta_0') + \lambda_0(2F_0'\theta_0).$$

Inspection of this equation suggests that θ_1 be written as

$$\theta_1 = \beta_1 B_1 + \lambda_0 L_0. \tag{21}$$

where B_1 and L_0 are functions of η which satisfy the differential equations

$$\left.\begin{aligned} M_1(B_1) &= -3f_1\theta_0', \qquad B_1(0) = B_1(\infty) = 0, \\ M_1(L_0) &= 2F_0'\theta_0, \qquad L_0(0) = L_0(\infty) = 0. \end{aligned}\right\} \tag{22}$$

Since F_0, f_1, f_{11}, etc., depend only on β_0, it follows from (22) that the same is true of B_1 and L_0. These functions do not depend upon particular surface temperature data. Thus, they are universal in the same sense as are the velocity functions.

Next, consider θ_2 and its differential equation (20 b). When the right side is evaluated using the universal velocity functions (14) and equation (21) for θ_1, we find that the factors β_1^2, β_2, $\beta_1\lambda_0$, λ_0^2, and λ_1 appear. So we write

$$\theta_2 = \beta_1^2 B_{11} + \beta_2 B_2 + \beta_1\lambda_0 Z_{1,0} + \lambda_0^2 L_{00} + \lambda_1 L_1. \tag{23}$$

The differential equations for B_{11}, B_2, etc., are found from the usual method of comparing coefficients to be

$$\left.\begin{aligned} M_2(B_{11}) &= 2f_1' B_1 - 3f_1 B_1' - 5f_{11}\theta_0', \\ M_2(B_2) &= -5f_2\theta_0', \\ M_2(Z_{1,0}) &= 2f_1'(\theta_0 + L_0) - 3f_1 L_0' + 2F_0' B_1, \\ M_2(L_{00}) &= 2F_0' L_0, \\ M_2(L_1) &= 2F_0'\theta_0, \end{aligned}\right\} \tag{24}$$

where

$$B_{11}(0) = B_{11}(\infty) = Z_{1,0}(0) = Z_{1,0}(\infty) = \ldots = 0.$$

Again, we have universal functions in the sense noted above.

In this fashion we can continue to recast each θ_n as a linear combination of universal functions. Further listing of the universal functions and their corresponding differential equations is given in Appendix A. It is not at all surprising that there are considerably more universal functions involved in our present velocity–temperature problem than there were for the velocity problem alone (compare equations (21), (23) and Appendix A with equation (14)).

It is interesting to look at some of the properties of the universal functions. An inspection shows that the differential equation for any one of the functions B does not include any of the functions L and Z as input data. The functions B may therefore be computed independently of the functions L and Z (the converse is not true). Using the results of Sparrow (1958), it may be observed that the functions B are simply the universal functions for an isothermal wall subjected to the free stream velocity of equation (10). The functions L and Z are associated with the deviations of the wall temperature from a uniform level. This splitting of the problem is, of course, associated with the linearity of the energy equation.

The symbolism used for the universal functions follows a rational path first set down by Görtler. It permits an easy way of identifying any universal function with its coefficient as they appear in the linear combinations which compose the θ_n. For example, B_2 is multiplied by β_2, L_{01} by λ_0 and λ_1, B_{11} twice by β_1.

Series representation of wall temperature variation

We now investigate the wall temperature variation corresponding to the assumed series representation for the principal thermal function λ. We had written

$$\lambda = \frac{\nu T'_w}{U(T_w - T_\infty)} = \sum_{n=0}^{\infty} \lambda_n \xi^n. \tag{16}$$

For the velocity variation (10), the relationship between ξ and x is given by (12). Using these facts, equation (16) may be rewritten as

$$\frac{T'_w}{T_w - T_\infty} = x^m\left(\sum_{n=0}^{\infty} s_n x^{n(m+1)}\right)\left\{\sum_{p=0}^{\infty} \lambda_p\left[\frac{x^{m+1}}{\nu(m+1)}\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n(m+1)}\right]^p\right\}. \tag{25}$$

It is easily seen that the series for T_w which satisfies equation (25) has the form

$$T_w - T_\infty = \sum_{n=0}^{\infty} a_n x^{n(m+1)} \qquad (a_0 \neq 0). \tag{26}$$

For the two important special cases of flow with a forward cuspidal point ($m = 0$) and of flow around a symmetrical body with stagnation point ($m = 1$) equation (25) reduces to

$$T_w - T_\infty = \sum_{n=0}^{\infty} a_n x^n \qquad (a_0 \neq 0, \quad m = 0), \tag{26a}$$

$$T_w - T_\infty = \sum_{n=0}^{\infty} a_n x^{2n} \qquad (a_0 \neq 0, \quad m = 1). \tag{26b}$$

For particular problems, the wall temperature and free stream velocity variations will usually be given, i.e. a_n and s_n will be specified. It is necessary to determine the λ_n from these. The needed relationship may be evaluated from (25) and (26). The first two of these are written below; a more complete listing is given in Appendix B:

$$\lambda_0 = \frac{\nu(m+1)}{s_0 a_0} a_1, \tag{27a}$$

$$\lambda_1 = \frac{\nu^2(m+1)^2}{s_0^2 a_0}\left[2a_2 - \frac{a_1 s_1}{s_0} - \frac{a_1^2}{a_0}\right]. \tag{27b}$$

It is clear that the determination of the λ_n from the given data is simply an arithmetic process.

Heat transfer

When the wall temperature is specified, the quantity of greatest practical interest is the heat transfer. From Fourier's law, the local heat flux q at the wall is

$$q = -\left(k\frac{\partial T}{\partial y}\right)_{y=0}. \tag{28}$$

In terms of the variables of this analysis, the expression for q becomes

$$\frac{qx}{(T_w - T_\infty)k} = Re\left[\sum_{n=0}^{\infty} \theta'_n(0)\xi^{n-1/2}\right] \tag{29}$$

where Re is the Reynolds number Ux/ν, and $\theta'_n(0)$ is an abbreviation for $[d\theta_n/d\eta]_0$. The $\theta'_n(0)$ may then be replaced by the universal functions in (21), (23) and Appendix A.

Variable fluid properties

Both Görtler's analysis and the one given here apply, with no essential change, to fluids with variable properties under the following circumstances: (1) $\rho\mu$ = constant, (2) ρk = constant (k = thermal conductivity), (3) c_p = constant. Then, using Howarth's transformation

$$Y = \int_0^y (\rho/\rho_\infty)\, dy, \tag{30}$$

and replacing y by Y in the equation (3) defining η, the entire formulation of the solution remains as before.

Concluding remarks

We have given a formally exact solution for the temperature problem under rather general variations of wall temperature and of free stream velocity. As a consequence of the introduction of universal functions, application of the results to particular situations becomes an arithmetical procedure. The practical utility of the development rests upon the availability of the universal functions. At present, only the functions B for $Pr = 1$ which correspond to the important cases of $m = 0$ and $m = 1$ are available (Sparrow 1958). Efforts are being made to find high speed, large memory computing equipment on which to evaluate a larger group of universal functions.

Appendix A. Universal functions and their differential equations

The universal functions are introduced by the following equations:

$$\theta_1 = \beta_1 B_1 + \lambda_0 L_0,$$

$$\theta_2 = \beta_1^2 B_{11} + \beta_2 B_2 + \beta_1 \lambda_0 Z_{1,0} + \lambda_0^2 L_{00} + \lambda_1 L_1,$$

$$\theta_3 = \beta_1^3 B_{111} + \beta_1 \beta_2 B_{12} + \beta_3 B_3 + \beta_1^2 \lambda_0 Z_{11,0} + \beta_1 \lambda_0^2 Z_{1,00} + \beta_1 \lambda_1 Z_{1,1} + \beta_2 \lambda_0 Z_{2,0} + \lambda_2 L_2 + \lambda_0 \lambda_1 L_{01} + \lambda_0^3 L_{000},$$

$$\theta_4 = \beta_1^4 B_{1111} + \beta_1^2 \beta_2 B_{112} + \beta_1 \beta_3 B_{13} + \beta_2^2 B_{22} + \beta_4 B_4 + \beta_1^3 \lambda_0 Z_{111,0} + \beta_1^2 \lambda_0^2 Z_{11,00} + \beta_1^2 \lambda_1 Z_{11,1} + \beta_1 \beta_2 \lambda_0 Z_{12,0} + \beta_1 \lambda_2 Z_{1,2} + \beta_1 \lambda_0 \lambda_1 Z_{1,01} + \beta_1 \lambda_0^3 Z_{1,000} + \beta_2 \lambda_0^2 Z_{2,00} + \beta_2 \lambda_1 Z_{2,1} + \beta_3 \lambda_0 Z_{3,0} + \lambda_3 L_3 + \lambda_0 \lambda_2 L_{02} + \lambda_0^2 \lambda_1 L_{001} + \lambda_1^2 L_{11} + \lambda_0^4 L_{0000}.$$

Using the $M_n(y)$ operator defined by (19), the differential equations for the universal functions are as follows.

Case $n = 1$. See (22).

Case $n = 2$. See (24).

Case $n = 3$.

$$M_3(B_{111}) = 4f_1' B_{11} - 3f_1 B_{11}' + 2f_{11}' B_1 - 5f_{11} B_1' - 7f_{111} \theta_0',$$

$$M_3(B_{12}) = 4f_1' B_2 - 3f_1 B_2' + 2f_2' B_1 - 5f_2 B_1' - 7f_{12} \theta_0',$$

$$M_3(B_3) = -7f_3 \theta_0',$$

$$M_3(Z_{11,0}) = 2F_0' B_{11} + 2f_1'(B_1 + 2Z_{1,0}) - 3f_1 Z_{1,0}' + 2f_{11}'(\theta_0 + L_0) - 5f_{11} L_0',$$

$$M_3(Z_{1,00}) = 2F_0' Z_{1,0} + 2f_1'(L_0 + 2L_{00}) - 3f_1 L_{00}',$$

$$M_3(Z_{1,1}) = 2F_0' B_1 + 2f_1'(\theta_0 + 2L_1) - 3f_1 L_1',$$

$$M_3(Z_{2,0}) = 2F_0' B_2 + 2f_2'(\theta_0 + L_0) - 5f_2 L_0',$$

$$M_3(L_2) = 2F_0' \theta_0, \quad M_3(L_{01}) = 2F_0'(L_0 + L_1), \quad M_3(L_{000}) = 2F_0' L_{00}.$$

Case $n = 4$.

$$M_4(B_{1111}) = 6f_1' B_{111} - 3f_1 B_{111}' + 4f_{11}' B_{11} - 5f_{11} B_{11}' + 2f_{111}' B_1 - 7f_{111} B_1' - 9f_{1111} \theta_0',$$

$$M_4(B_{112}) = 6f_1' B_{12} - 3f_1 B_{12}' + 4f_{11}' B_2 - 5f_{11} B_2' + 2f_{12}' B_1 - 7f_{12} B_1' + 4f_2' B_{11} - 5f_2 B_{11}' - 9f_{112} \theta_0',$$

$$M_4(B_{13}) = 6f_1' B_3 - 3f_1 B_3' - 9f_{13} \theta_0' + 2f_3' B_1 - 7f_3 B_1',$$

$$M_4(B_{22}) = 4f_2' B_2 - 5f_2 B_2' - 9f_{22} \theta_0',$$

$$M_4(B_4) = -9f_4 \theta_0',$$

$$M_4(Z_{111,0}) = 2F_0' B_{111} + 2f_1'(B_{11} + 3Z_{11,0}) - 3f_1 Z_{11,0}' + 2f_{11}'(B_1 + 2Z_{1,0}) - 5f_{11} Z_{1,0}' + 2f_{111}'(\theta_0 + L_0) - 7f_{111} L_0',$$

$$M_4(Z_{11,00}) = 2F_0' Z_{11,0} + 2f_1'(Z_{1,0} + 3Z_{1,00}) - 3f_1 Z_{1,00}' + 2f_{11}'(L_0 + 2L_{00}) - 5f_{11} L_{00}',$$

$$M_4(Z_{11,1}) = 2F_0' B_{11} + 2f_1'(B_1 + 3Z_{1,1}) - 3f_1 Z_{1,1}' + 2f_{11}'(\theta_0 + 2L_1) - 5f_{11} L_1',$$
$$M_4(Z_{12,0}) = 2F_0' B_{12} + 2f_1'(B_2 + 3Z_{2,0}) - 3f_1 Z_{2,0}' + 2f_{12}'(\theta_0 + L_0) - 7f_{12} L_0' + {} + 2f_2'(B_1 + 2Z_{1,0}) - 5f_2 Z_{1,0}',$$
$$M_4(Z_{1,2}) = 2F_0' B_1 + 2f_1'(\theta_0 + 3L_2) - 3f_1 L_2',$$
$$M_4(Z_{1,01}) = 2F_0'(Z_{1,0} + Z_{1,1}) + 2f_1'(L_0 + L_1 + 3L_{01}) - 3f_1 L_{01}',$$
$$M_4(Z_{1,000}) = 2F_0' Z_{1,00} + 2f_1'(L_{00} + 3L_{000}) - 3f_1 L_{000}',$$
$$M_4(Z_{2,00}) = 2F_0' Z_{2,0} + 2f_2'(L_0 + 2L_{00}) - 5f_2 L_{00}',$$
$$M_4(Z_{2,1}) = 2F_0' B_2 + 2f_2'(\theta_0 + 2L_1) - 5f_2 L_1',$$
$$M_4(Z_{3,0}) = 2F_0' B_3 + 2f_3'(\theta_0 + L_0) - 7f_3 L_0', \qquad M_4(L_3) = 2F_0' \theta_0,$$
$$M_4(L_{02}) = 2F_0'(L_2 + L_0), \qquad M_4(L_{001}) = 2F_0'(L_{00} + L_{01}),$$
$$M_4(L_{11}) = 2F_0' L_1, \qquad M_4(L_{0000}) = 2F_0' L_{000}.$$

Appendix B. Determination of λ_n for given s_n and a_n

Let
$$\Lambda_n = \lambda_n a_0 \left[\frac{s_0}{\nu(m+1)}\right]^{n+1},$$
then
$$\Lambda_0 = a_1,$$
$$\Lambda_1 = 2a_2 - a_1\left(\frac{s_1}{s_0} + \frac{a_1}{a_0}\right),$$
$$\Lambda_2 = 3a_3 - 3a_2\left(\frac{s_1}{s_0} + \frac{a_1}{a_0}\right) - a_1\left(\frac{s_2}{s_0} - \frac{3}{2}\frac{s_1^2}{s_0^2}\right) + \frac{a_1^2}{a_0}\left(\frac{3}{2}\frac{s_1}{s_0} + \frac{a_1}{a_0}\right)$$
$$\Lambda_3 = 4a_4 - 4a_3\left(\frac{3}{2}\frac{s_1}{s_0} + \frac{a_1}{a_0}\right) + a_2\left(5\frac{s_1^2}{s_0^2} - \frac{8}{3}\frac{s_2}{s_0} + 6\frac{a_1}{a_0}\frac{s_1}{s_0} - 2\frac{a_2}{a_0} + 4\frac{a_1^2}{a_0^2}\right) + {} + a_1\left(\frac{10}{3}\frac{s_1 s_2}{s_0^2} - \frac{s_3}{s_0} - \frac{5}{2}\frac{s_1^3}{s_0^3}\right) + \frac{a_1^2}{a_0}\left(\frac{4}{3}\frac{s_2}{s_0} - \frac{5}{2}\frac{s_1^2}{s_0^2} - 2\frac{a_1}{a_0}\frac{s_1}{s_0} + \frac{a_1^2}{a_0^2}\right).$$

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